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A Remark on p-Radical Groups

SHIGEO KOSHITANI*

Department of Mathematics, Faculty of Science, Chiba University, Yayoi-cho, Chiba City, 260, Japan

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1. INTRODUCTION

Let FG be the group algebra of a finite group G over an algebraically closed field F of characteristic $p>0$, and let P be a Sylow p-subgroup of G. We call G p-radical if the induced module $(F_P)^G$ from the trivial FP-module F_P is semi-simple (completely reducible) as a right FG-module. This definition is the same as that of Motose and Ninomiya in [7] (see [2, VI Corollary 6.4]).

The purpose of this note is to give one sufficient condition with respect to the Green correspondences under which G is p-radical. Namely,

THEOREM. *If the restriction $S_{N_G(Q)}$ of S to $F[N_G(Q)]$ is a simple $F[N_G(Q)]$-module for any simple FG-module S where Q is a vertex of S, then G is p-radical.*

The condition in the theorem is not too strong, for the following reasons. Assume that G is p-radical. Let Q be any p-subgroup of G and let $N=N_G(Q)$. Then the theorems of Okuyama [8] and Alperin [1] say that the Green correspondence with respect to (G, Q, N) gives a bijection between the set of all non-isomorphic simple FG-modules with vertex Q and the set of all non-isomorphic simple FN-modules with vertex Q (see Proposition 1). Let S be a simple FG-module with vertex Q. Then the above fact means that the restriction S_N of S to FN is simple modulo a kind of projectives. Moreover, if Q is a Sylow p-subgroup of G, then S_N is

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actually simple (see Proposition 2). Several years ago, Tsushima wrote a paper on p-radical groups [10]. The examples in his paper [10, Examples (1) and (2), p. 80] satisfy the condition in the theorem.

Throughout this paper, G is always a finite group and F is an algebraically closed field of characteristic $p > 0$, and we mean by an FG-module only a finitely generated right FG-module. We write $\dim V$ for the F-dimension of a vector space V over F. Let M and N be FG-modules. We write $\operatorname{Ker} M$ for the kernel of M (see [2, p. 86]) and $\operatorname{Soc} M$ for the socle of M as an FG-module. For a positive integer n, nM denotes the direct sum $M \oplus \cdots \oplus M$ (n times). We write $[M, N]_G$ for $\dim[\operatorname{Hom}_{FG}(M, N)]$. We write $N \mid M$ if N is (isomorphic to) a direct summand of M. Let H be a subgroup of G and L an FH-module. Then L^G denotes the induced FG-module from L, say, $L^G = L \otimes_{FH} FH$, and M_H denotes the restriction of M to FH. We write F_G for the trivial FG-module. For ordinary characters χ and ψ of G, $(\chi, \psi)_G$ denotes the inner product in G. We write 1_G for the trivial ordinary character of G. We write $k(G)$ for the number of all conjugacy classes of G. The symbol ▌ means that a proof is finished. We write $O^{p'}(G)$ for the intersection of all normal subgroups of G of index prime to p. The notations $|G|$, $|G:H|$, H^x, $Z(G)$, $N_G(X)$, $C_G(X)$, $H \lhd G$, $\langle x \rangle$, and $O_{p',p}(G)$ are standard (see Gorenstein [3, p. 511]). Other notation and notions follow the books of Feit [2] and Landrock [6].

2. Proof of the Theorem

In order to prove our main result, we proceed by a sequence of several lemmas.

Lemma 1. *Let $H \lhd G$ such that H is a p-group or p'-group. Assume that B and $\bar{B}$ are p-blocks of G and G/H, respectively, such that $\bar{B} \subseteq B$ and that $Q \subseteq \operatorname{Ker} S$ for any simple FG-module S in B where Q is a vertex of S. Then $V \subseteq \operatorname{Ker} T$ for any simple $F(G/H)$-module T in $\bar{B}$ where V is a vertex of T.*

Proof. This follows from [4, Lemma 1.3(c)]. ▌

Lemma 2 (Knörr). *If a vertex of S is contained in* $\operatorname{Ker} S$ *for any simple FG-module S in the principal p-block, then G is p-solvable.*

Proof. Let P be a Sylow p-subgroup of G. We may assume $P \neq 1$. Let S be any simple FG-module with vertex Q in the principal p-block. Then, by the theorem of Knörr [5, 3.7 Corollary (ii)], we may assume that $Z(P) \subseteq C_P(Q) \subseteq Q \subseteq P$. Hence, [2, IV Lemma 4.12(iii)] and the assumption imply that $1 \neq Z(P) \subseteq O_{p',p}(G)$. Thus, $G/O_{p',p}(G)$ is p-solvable from Lemma 1, [2, V Lemma 4.1], and induction. ▌

LEMMA 3 (Okuyama). *Let S be a simple FG-module with vertex Q, and let fS be the Green correspondent of S with respect to $(G, Q, N_G(Q))$. Then fS is simple if and only if $S \mid (F_Q)^G$.*

Proof. Let $N = N_G(Q)$ and $T = fS$. Assume that T is simple. Then, by [2, III Corollary 4.13], T is a simple-projective $F(N/Q)$-module. Hence $T \mid F(N/Q) \cong (F_Q)^N$, so that $S \cong f^{-1}T \mid T^G \mid (F_Q)^G$. The converse is proved by Okuyama [8, Lemma 2.2]. ∎

LEMMA 4. *Let S be a simple FG-module with vertex Q, and let $N = N_G(Q)$. Then the following are equivalent.*

(1) *S_N is a simple FN-module.*

(2) *S is a trivial source module and $Q \subseteq \operatorname{Ker} S$.*

Proof. (1) $\Rightarrow$ (2): Easy by Lemma 3 and [2, III Corollary 2.13(i)].

(2) $\Rightarrow$ (1): Let fS be the Green correspondent of S with respect to (G, Q, N). Then $S_N = fS \oplus (\bigoplus Y_i)$ for indecomposable FN-modules Y_i such that Y_i is $\mathfrak{y}$-projective for all i where $\mathfrak{y} = \{A \mid A \text{ is a subgroup of } N \cap Q^x \text{ for some } x \in G - N\}$. There exist no such Y_i's from [2, III Lemma 4.12] since $Q \subseteq \operatorname{Ker} S$. Hence, we obtain (1) by Lemma 3. ∎

LEMMA 5. *Let $H \lhd G$. If $S_{N_G(Q)}$ is simple for any simple FG-module S where Q is a vertex of S, then $T_{N_H(V)}$ is simple for any simple FH-module T where V is a vertex of T.*

Proof. Let T be a simple FH-module. Then, by Clifford theory and Frobenius reciprocity, $T \mid S_H$ for some simple FG-module S. Let Q be a vertex of S. By Lemma 4 and Mackey decomposition, T is a trivial source module and $V \subseteq \operatorname{Ker} S$ for a vertex V of T. Hence, Lemma 4 implies the assertion. ∎

The next lemma is a key of this paper. If G is p-radical, then G is p-solvable by Okuyama [9, Theorem 1]. Moreover, [7, Theorem 5] and inductive argument often make it possible to assume that G has a normal subgroup H of index p if we want to claim that G is p-radical. In general, G cannot be p-radical even if H is p-radical (see [7, Remark 2]). From this point of view, troubles come in when we claim that G would be p-radical.

LEMMA 6. *Let $H \lhd G$ and $|G : H| = p$. If H is p-radical and if $S_{N_G(Q)}$ is simple for any simple FG-module S where Q is a vertex of S, then G is also p-radical.*

Proof. Since G/H is cyclic and F is algebraically closed, we can assume from the theorem of Clifford that $\{S_i \mid i = 1, ..., n\}$ and $\{T_{ij} \mid i = 1, ..., m;$

$j=1, ..., p\} \cup \{T_i \mid i=m+1, m+2, ..., n\}$ are the sets of all non-isomorphic simple FG- and FH-modules, respectively, for integers m and n with $m<n$ such that $(S_i)_H \cong \bigoplus_{j=1}^{p} T_{ij}$ and $T_{i1}, ..., T_{ip}$ are all conjugate in G for each $i \leqslant m$, and that $(S_i)_H \cong T_i$ for each $i \geqslant m+1$. Let J be the Jacobson radical of FG. By Frobenius reciprocity, $(T_{ij})^G \cong S_i$ for each $i \leqslant m$ and each j, and $\operatorname{Soc}((T_i)^G) \cong (T_i)^G/(T_i)^G \cdot J \cong S_i$ for each $i \geqslant m+1$. Let P be a Sylow p-subgroup of G. Since H is p-radical, we can write

$$(F_{P \cap H})^H \cong \left(\bigoplus_{\substack{i \leqslant m \\ j}} a_{ij} T_{ij} \right) \oplus \left(\bigoplus_{i \geqslant m+1} a_i T_i \right)$$

for positive integers a_{ij} and a_i.

Fix any $i \leqslant m$. By Frobenius reciprocity and Mackey decomposition, $[(F_P)^G, S_i]_G = [(F_P)^G, (T_{ij})^G]_G = [((F_P)^G)_H, T_{ij}]_H = [(F_{P \cap H})^H, T_{ij}]_H = a_{ij}$ for all j.

Next, fix any $i \geqslant m+1$. Let $S=S_i$, $T=S_H$, $a=a_i$, and $b=[(F_P)^G, S]_G$. By Lemma 4 and the assumption, S is a trivial source module. Clearly, T is also a trivial source module. Let (F, R, K) be a p-modular system (see [6, p. 47]). Recall that F is algebraically closed. Then S is liftable to an R-free trivial source RG-module $\tilde{S}$ by [6, II Theorem 12.4(ii)]. Let χ be the K-character of G afforded by $\tilde{S} \otimes_R K$. Therefore, by Frobenius reciprocity and the result of Landrock [6, II Lemma 12.6(i)], $a=[(F_{P \cap H})^H, T]_H = [F_{P \cap H}, T_{P \cap H}]_{P \cap H} = \dim[\operatorname{Soc}(T_{P \cap H})] = (\chi, 1_{P \cap H})_{P \cap H} = (1/|P \cap H|) \cdot \sum_{y \in P \cap H} \chi(y)$. Similarly, $b=(\chi, 1_P)_P = (1/|P|) \cdot \sum_{y \in P} \chi(y)$. From the assumption, Lemma 4, and [2, III Lemma 4.12], for each $y \in G$, y is in some vertex of S if and only if y is a p-element in $\operatorname{Ker} S$. Let Y be the set of all such y's. Take any $y \in Y$. Then $S_{\langle y \rangle} \cong (\dim S) \cdot F_{\langle y \rangle}$, so that $\chi(y) = \dim S$ by the result of Scott [6, II Lemma 12.6(ii)]. Moreover, $\chi(z)=0$ for any $z \in P$ such that $z \notin Y$ by [6, II Lemma 12.6(iii)]. Therefore, $b=|P \cap \operatorname{Ker} S| \cdot (\dim S)/|P|$. Similarly, we have $a=|P \cap H \cap \operatorname{Ker} S| \cdot (\dim S)/|P \cap H|$. Let Q and V be vertices of S and T, respectively. Since G is p-solvable by the assumption and Lemmas 2 and 4, $|Q|/p = |V|$ by [4, Theorem 2.1]. Thus, $P \cap \operatorname{Ker} S \neq P \cap H \cap \operatorname{Ker} S$ by the assumption, Lemma 4, and [2, III Lemma 4.12]. Hence, $|P \cap \operatorname{Ker} S|/|P \cap H \cap \operatorname{Ker} S| = p$. Thus, $a=b$, so that $[(F_P)^G, S_i]_G = a_i$ for all $i \geqslant m+1$. This implies

$$(F_P)^G/(F_P)^G \cdot J \cong \left(\bigoplus_{i \leqslant m} a_{i1} S_i \right) \oplus \left(\bigoplus_{i \geqslant m+1} a_i S_i \right).$$

On the other hand, $\dim[(F_P)^G] = \dim[(F_{P \cap H})^H] = \sum_{i \leqslant m} a_{i1} (\sum_{j=1}^{p} \dim T_{ij}) + \sum_{i \geqslant m+1} a_i (\dim T_i) = \sum_{i \leqslant m} a_{i1} (\dim S_i) + \sum_{i \geqslant m+1} a_i (\dim S_i)$, which shows $(F_P)^G \cdot J = 0$. ∎

Now, we have come to a position to prove our main result.

Proof of Theorem. If G has a normal subgroup of index p, then G is p-radical by Lemmas 5 and 6 and induction. So, we may assume $G \neq O^{p'}(G)$ since G is p-solvable by Lemmas 2 and 4. Hence, G is p-radical by Lemma 5, [7, Theorem 5], and induction. ∎

EXAMPLES. The examples of p-radical groups mentioned in Tsushima's paper [10, Examples (1) and (2), p. 80], say, finite groups which have normal Sylow p-subgroups and the symmetric groups on 3 and 4 letters for all primes p, satisfy the condition in the Theorem.

3. PROPOSITIONS

In this section, we state two propositions and one corollary, which are mentioned in the introduction.

PROPOSITION 1 (Okuyama and Alperin). *Assume that G is p-radical. Let Q be any p-subgroup of G and let $N = N_G(Q)$. Then the Green correspondence with respect to (G, Q, N) gives a bijection between the set of all non-isomorphic simple FG-modules with vertex Q and the set of all non-isomorphic simple FN-modules with vertex Q.*

Proof. This follows from the properties of the Green correspondence [2, III Theorem 5.6], the theorems of Okuyama [8, Lemma 2.2] and Alperin [1, Lemma 1], [2, III Corollary 4.13], and the definition of p-radical groups. ∎

PROPOSITION 2 (Okuyama). *Let P be a Sylow p-subgroup of G, and let $N = N_G(P)$. Assume that G is p-radical. Then we have the following.*

(i) *If S is a simple FG-module such that P is a vertex of S, then S_N is a simple FN-module and $O^{p'}(G) \subseteq \operatorname{Ker} S$.*

(ii) $k(G/O^{p'}(G)) = k(N/P)$.

Proof. (i) follows from the result of Okuyama [9, Lemma 1], Lemma 4, and Sylow's theorem. (ii) is clear by (i), Proposition 1, and [2, III Corollary 2.13(i) and Lemma 4.12]. ∎

COROLLARY. *Assume that G is p-radical and that $G = O^{p'}(G)$. Then we have the following.*

(i) *If S is a simple FG-module whose vertex is a Sylow p-subgroup of G, then S is the trivial FG-module F_G.*

(ii) *For a Sylow p-subgroup P of G, $N_G(P) = P$.*

Proof. (i) and (ii) follow from Proposition 2(i) and Proposition 2(ii), respectively. ▮

Remark. In the corollary, the assumption that $G = O^{p'}(G)$ is not essential when we investigate the structure of G such that G is p-radical because of [7, Theorem 5] (see the remark just before Lemma 6).

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